Introduction

The methods developped in "Lokal präsentierbare Kategorien" (L.N. vol. 221) were not sufficient to decide whether any of the following categories were locally presentable: the category of functors on a small category $\underline{U}$ with values in a locally presentable category which preserve a given class of colimits in $\underline{U}$, the category of cosheaves on a site with values in a locally presentable category, the categories of coalgebras, bialgebras, Hopfalgebras ... over a commutative ring Λ and likewise the category of comodules (resp. bimodules) over a Λ-coalgebra (resp. Λ-bialgebra), the category $\underline{A}_{\mathbb{G}}$ of $\mathbb{G}$-coalgebras, where $\mathbb{G}$ is a cotriple with rank in a locally presentable category $\underline{A}$, the category $\mathrm{Adj}(\underline{A},\underline{B})$ of adjoint functors between two locally presentable categories $\underline{A}$ and $\underline{B}$, etc. These questions were solved affirmatively in [31], [32], [33] and [34] by new techniques. In the process the notion of a bialgebra in a category - generalizing the notion of a bialgebra over a commutative ring - emerged as the unifying concept from the point of view of the constructions on which the proofs were based. The basic problem in all cases involved the construction of generators in the category under consideration which in turn lead to the following general question: *Given an object* A *in a category* $\underline{A}$ *equipped with a structure* $\mathcal{H}$ *and given a subobject* U *of* A *in* $\underline{A}$. *How does one construct a subobject* U' *with structure* $\mathcal{H}'$ *containing* U *such that the inclusion* $U' \hookrightarrow A$ *is compatible with the structures* $\mathcal{H}'$ *and* $\mathcal{H}$ *and such that* U' *is not much bigger than* U ? The complexity of this problem is perhaps best illustrated by two seemingly unrelated examples: Given a Hopfalgebra H over a commutative ring Λ and a Λ-submodule U of H . Find a sub-Hopfalgebra U' of H containing U such that the underlying Λ-module of U' is not much bigger than U ; or more specifically, that the size of U' depends only on U but not on H. Clearly U' is - if it exists - not unique because there is no such thing as "the" sub-Hopfalgebra "generated" by U. (For coalgebras over a commutative ring the corresponding problem was investigated by M.Barr [1] using purity.) On the other hand consider an object A equipped with a descent datum φ_A and a subobject U of A. Find a subobject U' containing U and a descent datum $\varphi_{U'}$ on U' such that $\varphi_{U'}$ is compatible with φ_A and U' is not much bigger than U. A construction of $(U', \varphi_{U'})$ was given by Grothendieck and Verdier in SGA 4 (p. 138-179) in a more general context. But the proof has a gap and their size estimate of U' is false.

Our main results consist in 1) making precise what an object with structure is - this is done by the notion of a bialgebra in a category 2) solving the above mentioned problem for bialgebras in locally presentable categories under appropriate conditions and 3) establishing size estimates for the constructed sub-bialgebras which in most cases are the best possible (cf. 3.1, 3.8, 3.22). With the exception of §5 all our results in §3 - §6 are applications of this.

Roughly speaking a bialgebra in a category $\underline{A}$ with respect to a given set M of operations and a set R of relations consists of an object $A \in \underline{A}$ together with a structure morphism μ_A for every $\mu \in M$ and a functorial diagramm for every $r \in R$ which commutes. In the literature so far a structure morphism μ_A on an object A is a morphism like $A \times A \longrightarrow A$, $A \longrightarrow A \coprod A$, $A \otimes A \longrightarrow A$, $A \longrightarrow A$, $A \otimes A \longrightarrow A \otimes A$, etc. In contrast we allow it to be a morphism $FA \longrightarrow F'A$, where F and F' is any pair of functors with domain $\underline{A}$ and a common codomain (the latter can depend on μ). F is called the domain of μ and F' the codomain. Likewise a relation is normally given by diagrams such as $A \times \ldots \times A \rightrightarrows A$, $A \rightrightarrows A \coprod \ldots \coprod A$, $A \otimes \ldots \otimes A \rightrightarrows A$, $A \rightrightarrows A \otimes \ldots \otimes A$, $A \rightrightarrows A$, $A \otimes \ldots \otimes A \rightrightarrows A \otimes \ldots \otimes A$, etc. which are built up of structure morphisms $\mu_A, \mu \in M$, and canonical morphisms (like twisting, etc.)

The relevant aspect here is that these diagrams are natural with respect to those morphisms $A \longrightarrow A'$ in $\underline{A}$ which are compatible with the given operations. Therefore we define a relation r to be a map which assigns to every object A equipped with structure morphisms μ_A, $\mu \in M$, a diagram $GA \rightrightarrows G'A$ which is natural in the sense just mentioned and where G and G' is any pair of functors with domain $\underline{A}$ and common codomain (the latter can depend on r). An object equipped with structure morphisms is said to satisfy the relation r if the corresponding diagram commutes. In this way one obtains the category $\mathrm{Bialg}(\underline{A})$ of bialgebras in $\underline{A}$ with respect to specified operations M and relations R. The morphisms in $\mathrm{Bialg}(\underline{A})$ are those morphisms in $\underline{A}$ which are compatible with the operations in M.

The notion of a bialgebra covers a wide range of examples, e.g. universal algebras resp. coalgebras in a category with finite products resp. coproducts in the sense of Lawvere [21] or Birkhoff [2], coalgebras over an arbitrary commutative ring Λ and likewise Λ-Hopfalgebras resp. Λ-bialgebras in the usual sense (more generally tensor product preserving functors on a Prop in the sense of MacLane [24]), comodules over a Λ-coalgebra, bimodules over a Λ-bialgebra, algebras over a triple, coalgebras over a cotriple, données de recollements, descent data and more generally sections (resp. cartesian closed sections) with respect to a fibration, functors on a small category which preserve a given class of limits resp. colimits, sheaves and cosheaves on a site, pairs of adjoint functors between locally presentable categories and more generally Σ-continuous resp. Σ-cocontinuous functors on a small category $\underline{U}$ with respect to an arbitrary class Σ of morphisms in the set valued functor category $[\underline{U}^0, \underline{\mathrm{Sets}}]$, and finally Σ-closed objects in a category $\underline{A}$ with respect to a bifunctor $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ and a class Σ of morphisms in $\underline{B}$ (Recall that $A \in \underline{A}$ is called Σ-closed with respect to T if

$T(\sigma,A)$ is an isomorphism for every $\sigma \in \Sigma$).

A bialgebra in a category $\underline{A}$ is denoted with (A,M,R), where $A \in \underline{A}$ is the underlying object and M and R refer to the specified operations and relations. Given a subobject U of the underlying object A we are concerned with the construction of sub-bialgebras (U',M,R) of (A,M,R) containing U such that U' is "as small as possible", in particular the construction should provide effective size estimates for U' in terms of U, M and R (but not A). More generally given a bialgebra (A,M,R) and an object U we investigate factorizations of a morphism $f : U \longrightarrow A$ into a morphism $U \to U'$ and a bialgebra morphism $(U',M,R) \longrightarrow (A,M,R)$ such that U' is not much bigger than U and its size can be estimated in terms of U, M and R. It is obvious that without conditions on M, R and on the underlying category $\underline{A}$ no reasonable answers can be expected. In order to elaborate on these conditions we recall a few basic facts about locally α-presentable categories.

Let $\alpha \geq \aleph_0$ be a regular cardinal. A directed set is called α-filtered if every subset with less than α elements has an upper bound. A functor F is said to preserve α-filtered colimits if the domain of F has colimits over α-directed sets and F preserves them. An object A in a category $\underline{A}$ is called α-presentable (resp. α-generated) if the hom functor $[A,-]: \underline{A} \longrightarrow \underline{\text{Sets}}$ preserves α-filtered colimits (resp. preserves those α-filtered colimits whose transition morphisms are monomorphic). For instance, if $\underline{A}$ is the category of groups, rings, modules over a ring, etc., then $A \in \underline{A}$ is α-presentable (resp. α-generated) iff A admits a presentation in the usual sense by <u>less</u> than α generators and <u>less</u> than α relations (resp. <u>less</u> than α generators). In particular $\aleph_0$-presentable (resp. $\aleph_0$-generated) is equivalent with finitely presentable (resp. finitely generated) and likewise $\aleph_1$-presentable (resp. $\aleph_1$-generated) with countably presentable (resp.

countably generated). A category $\underline{A}$ is called locally α-presentable if it has colimits (i.e. sums and cokernels) and a set M of α-presentable generators. (It is called locally presentable if it is locally α-presentable for some α.) In a locally α-presentable category $\underline{A}$ every object is α'-presentable for some regular cardinal α' and, roughly speaking, for $\beta \geq \alpha$ an object $A \in \underline{A}$ is β-presentable iff it is the cokernel of two morphisms $\coprod_{j\in J} U_j \rightrightarrows \coprod_{i\in I} U_i$, where $U_i, U_j \in M$ and J and I have less than β elements. Moreover $\underline{A}$ has limits (= inverse limits), is cowellpowered and α-filtered direct limits commute with kernels and products with less than α factors. Also a functor F between locally presentable categories preserves γ-filtered colimits for some γ provided it has either a left or right adjoint. The class of locally presentable categories is larger than one might expect and includes the categories of sets, groups, rings, modules and more generally universal algebras, the category of set (group, ring ...) valued sheaves on a small category with respect to a Grothendieck topology, the category of set (group, ring ...) valued functors on a small category $\underline{U}$ which preserve a given set of limits in $\underline{U}$ (e.g. the category $\underline{\text{Cat}}$ of small categories and other "universal algebras" with partial operations), the dual category $\underline{\text{Comp}}^{\text{o}}$ of compact spaces, etc. In contrast the categories $\underline{\text{Comp}}$ and $\underline{\text{Top}}$ of (compact) topological spaces and other related categories are not locally presentable.

For the above mentioned construction of sub-bialgebras of a bialgebra containing a given subobject (resp. the decomposition of a morphism into a morphism and a bialgebra morphism) we need the following.

1) the underlying category $\underline{A}$ and the categories occuring in the definition of the operations and relations are locally presentable (or more generally "catégories localisables" in the sense of Y.Diers [5]).

2. the operations M and relations R form a set and the functors which are domain or codomain of either an operation or a relation

preserve β-filtered colimits for some cardinal β.

Then there are cardinals γ such that a bialgebra (X,M,R) is γ-presentable in $\mathrm{Bialg}(\underline{A})$ iff its underlying object X is γ-presentable in $\underline{A}$ (cf. 3.8). Moreover for a bialgebra (A,M,R) and a γ-presentable object $U \in \underline{A}$ every morphism $f : U \longrightarrow A$ admits a decomposition into a morphism $U \rightarrow U'$ and a bialgebra morphism $(U',M,R) \longrightarrow (A,M,R)$ such that U' is again γ-presentable (cf. 3.8). The class of all such γ's is cofinal in the class of all cardinals. Of special interest is the smallest possible γ . Estimates are given in terms of $\underline{A}$, M and R . (The analogue assertions concerning the existence and size estimates of sub-bialgebras containing a given subobject are discussed later on.) We illustrate the above with some examples.

a) For Hopfalgebras over a commutative ring Λ one can choose for γ any cardinal $\geq \aleph_1$ (cf. 4.4). In particular every Λ-homomorphism $U \longrightarrow H$ from a countably presentable Λ-module U to an arbitrary Λ-Hopfalgebra H admits a decomposition into a Λ-homomorphism $U \rightarrow U'$ and a Hopfalgebra morphism $U' \longrightarrow H$ such that the underlying Λ-module of U' is again countably presentable (the corresponding assertion for finitely presentable Λ-modules is obviously false). Moreover the Λ-Hopfalgebras whose underlying Λ-module is countably presentable form a "set" of dense generators in the category of Λ-Hopfalgebras (i.e. the equivalence classes of such Hopfalgebras form a set).
The same holds for Λ-bialgebras, Λ-coalgebras etc. (cf. 4.3-4.7). Moreover the following categories are locally $\aleph_1$-presentable: commutative Λ-Hopfalgebras, cocommutative Λ-Hopfalgebras, bicommutative Λ-Hopfalgebras, Λ-bialgebras, commutative Λ-bialgebras, cocommutative Λ-bialgebras, bicommutative Λ-bialgebras, Λ-coalgebras, cocommutative Λ-coalgebras, comodules over a Λ-coalgebra, bimodules over a Λ-bialgebra, etc (cf. 4.3-4.9).

b) Let $\mathfrak{F}$ be a fibration with base $\underline{C}$ and let $\alpha : S_o \longrightarrow S$ be a morphism in $\underline{C}$ such that 1) the fibres $\underline{\mathfrak{F}}_{S_o}$ and $\underline{\mathfrak{F}}_S$ over S_o and S are locally countably presentable categories and 2) the inverse image functors α^*, p_1^*, p_2^* and p_{31}^* preserve filtered colimits and take countably presentable objects into countably presentable objects (cf. Grothendieck [16], also for the notation). Then for an object $A \in \underline{\mathfrak{F}}_{S_o}$ with descent datum φ_A and a countably presentable object $U \in \underline{\mathfrak{F}}_{S_o}$ every morphism $f : U \longrightarrow A$ admits a factorization into a morphism $U \longrightarrow U'$ and a morphism $(U',\varphi_{U'}) \longrightarrow (A,\varphi_A)$ between descent data such that U' is again countably presentable (cf. 4.14, 4.15). As a consequence the category $\mathrm{Desc}(\underline{\mathfrak{F}}_{S_o})$ of descent data is locally $\aleph_1$-presentable and the forgetful functor $\mathrm{Desc}(\underline{\mathfrak{F}}_{S_o}) \longrightarrow \underline{\mathfrak{F}}_S$ cotripleable provided the inverse image functors preserve colimits (cf. 4.15). Likewise $\mathrm{Desc}(\underline{\mathfrak{F}}_{S_o})$ is a Grothendieck category (resp. a topos) provided the fibres are, and the inverse image functors preserve colimits and finite limits (4.16).
If $\alpha : S_o \longrightarrow S$ is of $\mathfrak{F}$-descent type (cf. Grothendieck [17] 1.7), then the above implies that every descent datum on objects of $\underline{\mathfrak{F}}_{S_o}$ is effective providdd every descent datum on countably presentable objects is effective (cf. 4.18).
Similar assertions hold for sections and cartesian closed sections with respect to a fibration (cf. 4.19-4.26).

c) Let $\mathbb{G} = (G,\varepsilon,\delta)$ be a cotriple in a locally α-presentable category $\underline{A}$ and assume that $G : \underline{A} \longrightarrow \underline{A}$ preserves β-filtered colimits for some β. Let $\gamma \geq \sup(\aleph_1,\alpha,\beta)$. Then for a $\mathbb{G}$-coalgebra (A,ξ) and a γ-presentable object $U \in \underline{A}$ every morphism $f : U \longrightarrow A$ admits a factorization into a morphism $U \longrightarrow U'$ and a $\mathbb{G}$-coalgebra morphism $(U',\xi') \longrightarrow (A,\xi)$ such that $U' \in \underline{A}$ is again γ-presentable (cf. 4.10). This implies that the category $\underline{A}_{\mathbb{G}}$ of $\mathbb{G}$-coalgebras is locally $\sup(\aleph_1,\alpha,\beta)$-presentable and that a $\mathbb{G}$-coalgebra is γ-presentable iff

its underlying object is. Moreover $\underline{A}_G$ is a topos (resp. a Grothendieck category) provided $\underline{A}$ is and $G : \underline{A} \longrightarrow \underline{A}$ preserves finite limits (cf. 4.11).
Applications of this are given for comodules over a Λ-coalgebra (cf. 4.8) and for bimodules over a Λ-bialgebra (cf. 4.9).

d) Let $\underline{U}$ be a small category and Σ a set of morphisms in $[\underline{U}^o, \underline{Sets}]$. Let $\underline{X}$ be a locally α-presentable category and let $Cc_\Sigma[\underline{U},\underline{X}]$ be the category of all Σ-cocontinuous functors. For instance if Σ is given by a set K of colimits in $\underline{U}$ (resp. by a Grothendieck topology τ) then the Σ-cocontinuous functors $\underline{U} \rightarrow \underline{X}$ are exactly the K-colimit preserving functors on $\underline{U}$ (resp. the τ-cosheaves on $\underline{U}$). Let γ be any regular cardinal such that $\alpha \leq \gamma \geq \aleph_1$ and $\gamma > \mathrm{card}(\Sigma)$, $\gamma > \mathrm{card}(d\sigma(U))$, $\gamma > \mathrm{card}(r\sigma(u))$ for every $\sigma \in \Sigma$ and $U \in \underline{U}$, where $d\sigma$ and $r\sigma$ denote the domain and codomain of σ. Then for a Σ-cocontinuous functor $t : \underline{U} \longrightarrow \underline{X}$ and a γ-presentable functor $s \in [\underline{U},\underline{X}]$ every natural transformation $s \longrightarrow t$ admits a decomposition $s \longrightarrow s' \longrightarrow t$ such that $s' : \underline{U} \longrightarrow \underline{X}$ is Σ-cocontinuous and again γ-presentable in $[\underline{U},\underline{X}]$. This implies that the category $Cc_\Sigma[\underline{U},\underline{X}]$ of Σ-cocontinuous functors is locally γ-presentable and that the inclusion $Cc_\Sigma[\underline{U},\underline{X}] \xrightarrow{\subset} [\underline{U},\underline{X}]$ has a right adjoint. The latter has been a long outstanding problem in category theory.
The above can be generalized to a class Σ of morphisms whose codomains $\{r\sigma | \sigma \in \Sigma\}$ form a set (modulo equivalence). Therefore we can also consider functors, which preserve a given class of colimits in $\underline{U}$ (in particular one can choose all existing colimits in $\underline{U}$). The above size estimates for γ however have to be replaced by more elaborate ones. The apparatus needed for the generalization to a class Σ is substantial (the entire chapter § 5 concerning purity and a good deal of § 6). Further generalizations concern the replacement of $\underline{X}$ by a topological category over $\underline{X}$ (cf. 6.21).

e) The category $\mathrm{Adj}(\underline{A},\underline{B})$ of adjoint fuctors between locally presentable categories $\underline{A}$ and $\underline{B}$ can be shown to be equivalent with the category of Σ-cocontinuous functors $\underline{U} \longrightarrow \underline{B}$ for an appropriate small category $\underline{U}$ and a set Σ of morphisms in $[\underline{U}^o, \underline{\mathrm{Sets}}]$ (cf.6.19). Thus by d) above $\mathrm{Adj}(\underline{A},\underline{B})$ is again locally presentable. In contrast if $\underline{A}$ and $\underline{B}$ are Grothendieck categories (or topoi), then $\mathrm{Adj}(\underline{A},\underline{B})$ need not be so. A surprising counter example is the following. Let $\underline{A}$ be the category of abelian p-groups for some prime p and $\underline{B} = \underline{\mathrm{Ab}}.\underline{\mathrm{Gr}}.$ the category of all abelian groups. Then $\mathrm{Adj}(\underline{A},\underline{B})$ can be shown to be equivalent with the category of p-adic complete abelian groups (cf. 6.25 c)).

f) Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor between locally presentable categories and let Σ be a set of morphisms in $\underline{B}$. Let $\underline{A}_{\Sigma,T}$ be the full subcategory of $\underline{A}$ consisting of all $X \in \underline{A}$ such that $T(\sigma,X)$ is an isomorphism for every $\sigma \in \Sigma$. For example T can be $\otimes_\Lambda$, $\mathrm{Tor}_n^\Lambda(-,-)$, $[-,-]$, $\mathrm{Ext}_\Lambda^n(-,-)$ etc. and Σ the inclusion of a set $\mathfrak{F}$ of right ideals in the ring Λ. Assume that for every $B \in \underline{B}$ there is a cardinal β_B such $T(B,-)$ preserve β_B-filtered colimits (which is obviously the case for the above examples). Then there are cardinals γ such that every morphism $f : U \longrightarrow A$ with $A \in \underline{A}_{\Sigma,T}$ and U γ-presentable in $\underline{A}$ admits a decomposition $U \rightarrow U' \longrightarrow A$ with $U' \in \underline{A}_{\Sigma,T}$ and U' being again γ-presentable in $\underline{A}$ (cf. 6.2). For instance if T is as above, $\mathfrak{F}$ is countable and the ideals $I \in \mathfrak{F}$ countably presentable, then one can choose for γ any cardinal $\geq \aleph_1$.

If $T = \otimes_\Lambda$, then $\underline{A}_{\Sigma,\otimes_\Lambda}$ consists of modules which are uniquely divisible by the ideals of $\mathfrak{F}$. For instance, let $\underline{A}$ and $\underline{B}$ be Grothendieck categories and $U \in \underline{A}$ a generator with endomorphism ring Λ. Then the category $\mathrm{Adj}(\underline{A},\underline{B})$ of adjoint functors between $\underline{A}$ and $\underline{B}$ is equivalent with the full subcategory of ${}_\Lambda\underline{B}$ consisting of those left Λ-objects which are uniquely divisible by the Gabriel filter $\mathfrak{F}$ in Λ associated with $\underline{A}$ (cf. 6.25 b)).

We now return to the problem of constructing sub-bialgebras (U',M,R) of a bialgebra (A,M,R) which contain a given subobject $U \subset A$ such that U' is not much bigger than U. This can be done under the conditions as above (cf. 1) and 2)) but the size estimates for U' are different, in general less effective. They are best stated in terms of noetherian conditions. The details are too involved to be given here (cf. 3.22, 3.23). and we illustrate them with an example. Let Λ be a commutative noetherian ring (or more generally a $\aleph_1$-noetherian ring which means that every countably generated ideal is countably presentable). Then every countably generated Λ-submodule of a Λ-Hopfalgebra is contained in a sub-Hopfalgebra whose underlying Λ-module is again countably generated. The same holds for Λ-bialgebras, Λ-coalgebras etc. If Λ is not $\aleph_1$-noetherian this need not be so. However there is always a cardinal γ such that Λ is γ-noetherian (i.e. every γ-generated ideal is γ-presentable). Then the above holds for γ-generated Λ-submodules of Λ-Hopfalgebras, etc. The same phenomenon happens for locally presentable categories. By Gabriel-Ulmer [13], 13.3 a locally α-presentable category is locally γ-noetherian for some $\gamma \geq \alpha$ The increase of γ over α accounts for the less effective size estimates for the constructed sub-bialgebras.

The basic idea for the construction of sub-bialgebras I got in a seminar of the University of Zurich 1974/75 in which Kaplansky's decomposition of projective modules into a direct sum of countably generated projective modules was studied (among other things). The parallel may be still apparent in § 1 in which an "elementwise" exposition of the basic techniques is given. The incentive to study sub-bialgebras "generated" by a subobject resulted from a problem which was given to us (= a group of students) in Heidelberg in 1964 by A.Dold. He suggested to investigate the category of cocontinuous abelian group valued functors on a Grothendieck category $\underline{A}$ in terms of a generator $U \in \underline{A}$ and its endomorphism ring Λ . We didn't get anywhere with it at the time but I kept it in the back of my mind and worked on it from time to time without much success. The turning point was the discovery that in the special case $\underline{A}$ = abelian p-groups the category of cocontinuous functor is equivalent with the category of p-adic complete abelian groups.