

§ 2 Review of locally presentable categories

2.1 Definition ([13] 5.1) Let  $\alpha \geq \aleph_0$  be a regular cardinal. A small category  $\underline{D}$  is called  $\alpha$ -filtered if

a) for every family  $(D_i)_{i \in I}$  of objects in  $\underline{D}$  with  $\text{card}(I) < \alpha$  there exists an object  $D \in \underline{D}$  together with a morphism  $D_i \rightarrow D$  for every  $i \in I$ .

b) for every family  $(D_i \xrightarrow{\gamma_i} D_1)_{i \in I}$  of morphisms in  $\underline{D}$  with  $\text{card}(I) < \alpha$  there exists a morphism  $\gamma : D_1 \rightarrow D_2$  such that  $\gamma \circ \gamma_i = \gamma \circ \gamma_j$  for every pair  $i, j \in I$ .

For  $\alpha = \aleph_0$  this specializes to the usual definition of filtered colimits (resp. direct limits).

A functor  $F : \underline{A} \rightarrow \underline{B}$  is said to preserve  $\alpha$ -filtered colimits if it preserves colimits over  $\alpha$ -filtered categories. The least regular cardinal  $\alpha$  with this property is called the presentation rank of  $F$  and denoted by  $\pi(F)$ . Examples are functors  $F : \underline{A} \rightarrow \underline{B}$  which have a right adjoint or - somewhat surprisingly - functors  $F : \underline{A} \rightarrow \underline{B}$  between locally presentable categories (2.3) which have a left adjoint, in particular underlying or forgetful functors (cf. 2.9, 3.4 c)).

Likewise a functor  $F : \underline{A} \rightarrow \underline{B}$  is said to preserve monomorphic  $\alpha$ -filtered colimits if it preserves colimits over  $\alpha$ -filtered categories whose transition morphisms in  $\underline{A}$  are monomorphic. (This does not mean that  $F$  preserves monomorphisms.) The least regular cardinal  $\alpha$  with this property is called the generation rank and denoted with  $\epsilon(F)$ .

2.2 Definition (cf [13] 6.1) Let  $\alpha \geq \aleph_0$  be a regular cardinal and let  $\underline{A}$  be a category with  $\alpha$ -filtered colimits. An object  $A \in \underline{A}$  is called  $\alpha$ -presentable (resp.  $\alpha$ -generated) if the hom-functor  $[A, -] : \underline{A} \rightarrow \underline{\text{Sets}}$  preserves  $\alpha$ -filtered colimits (resp. monomorphic  $\alpha$ -filtered colimits). The least regular cardinal  $\alpha \geq \aleph_0$  with this

property is called the presentation rank (resp. generation rank) and denoted with  $\pi(A)$  (resp.  $\varepsilon(A)$ ). Clearly  $\pi(A) \geq \varepsilon(A)$ .

It may appear that this definition is stronger than the one given in the introduction. This is however not the case, at least in practise. First by Swan every  $\aleph_0$ -filtered category admits a cofinal directed set. Hence for  $\alpha = \aleph_0$  the two notions coincide. Second for  $\alpha > \aleph_0$  the two definitions are equivalent in a locally  $\alpha$ -presentable category. Moreover they lead to the same notion of a locally  $\alpha$ -presentable category in 2.3 below. This can be shown by going over the proofs of § 7 in Gabriel-Ulmer [13].

2.3 Definition (cf [13] 7.1, 9.1) Let  $\alpha \geq \aleph_0$  be a regular cardinal. A category  $\underline{A}$  is called locally  $\alpha$ -presentable if  $\underline{A}$  has colimits and a set  $M$  of  $\alpha$ -presentable generators ( $M$  is a set of generators means: A morphism  $f : A \rightarrow A'$  is an isomorphism iff  $[U, f]$  is a bijection for every  $U \in M$ ).

Likewise a category  $\underline{A}$  is called locally  $\alpha$ -generated if  $\underline{A}$  has colimits and a set  $M$  of  $\alpha$ -generated generators such that every coproduct  $\coprod_{i \in I} U_i$  with  $U_i \in M$  and  $\text{card}(I) < \alpha$  has only a set of proper quotients. (Recall that an epimorphism  $p : X \rightarrow Y$  is called proper if it does not factor through a proper subobject of  $Y$ .) The least regular cardinal  $\alpha \geq \aleph_0$  with this property is called the presentation rank of  $\underline{A}$  (resp. the generation rank) and denoted with  $\pi(\underline{A})$  (resp.  $\varepsilon(\underline{A})$ ).

2.4 A category is called locally presentable (resp. locally generated) if it is locally  $\alpha$ -presentable (resp. locally  $\alpha$ -generated) for some  $\alpha$ .

2.5 A locally  $\alpha$ -presentable category is locally  $\alpha$ -generated ([13] 6.6c).

Surprisingly there is a converse: A locally  $\alpha$ -generated category is locally  $\beta$ -presentable for some  $\beta \geq \alpha$  (cf [13] 9.8, 9.10).

2.6 A locally  $\alpha$ -presentable category has limits ([13] 1.12) and is cowellpowered ([13] 7.14; i.e. every object has only a set of quotients).

Moreover  $\alpha$ -filtered colimits commute with  $\alpha$ -limits (cf [13] 7.12; recall that  $\varprojlim (D \xrightarrow{H} A)$  is called an  $\alpha$ -limit if  $D$  has less than  $\alpha$  morphisms, [13] § 0).

2.7 In a locally presentable category  $\underline{A}$  with a set  $M$  of  $\alpha$ -presentable generators an object  $A \in \underline{A}$  is  $\beta$ -generated for some  $\beta \geq \alpha$  iff there is a proper epimorphism  $\bigsqcup_{i \in I} U_i \rightarrow A$  with  $U_i \in M$  and  $\text{card}(I) < \beta$  (cf [13] 9.3). If moreover  $M$  is a set of regular generators, then  $A \in \underline{A}$  is  $\beta$ -presentable iff there is a cokernel diagram

$$\bigsqcup_{j \in J} U_j \rightrightarrows \bigsqcup_{i \in I} U_i \longrightarrow Y$$

with  $U_i, U_j \in M$  and  $\text{card}(J) < \beta < \text{card}(I)$  such that  $A$  is a retract of  $Y$  (cf [13] 7.6). (Recall that  $M$  is called regular if for every  $A \in \underline{A}$  there is a cokernel diagram  $K \rightrightarrows \bigsqcup_{v \in V} U_v \rightarrow A$  with  $U_v \in M$ .)

Moreover there is a regular cardinal  $\delta$  such that every  $\delta$ -generated object in  $\underline{A}$  is  $\delta$ -presentable and  $\delta$  can be chosen so as to exceed any given cardinal (cf [13] 13.3).

2.8 In a locally  $\alpha$ -presentable category  $\underline{A}$  the full subcategory  $\underline{A}(\alpha)$  of all  $\alpha$ -presentable objects is small and closed in  $\underline{A}$  under  $\alpha$ -colimits. The same holds for the full subcategory  $\tilde{\underline{A}}(\alpha)$  of all  $\alpha$ -generated objects ([13] 6.2). In particular for every  $A \in \underline{A}$  the category  $\underline{A}(\alpha)/A$  of  $\alpha$ -presentable objects over  $A$  is small and  $\alpha$ -filtered, and the colimit of the forgetful functor  $\underline{A}(\alpha)/A \rightarrow \underline{A}$ ,  $(U \rightarrow A) \rightsquigarrow U$ , is  $A$  (cf [13] 2.6, 7.4, 3.1). The same holds for the category of  $\alpha$ -generated subobjects of  $A$  (cf [13] 9.5). The functor

$$\underline{A} \longrightarrow [\underline{A}(\alpha)^0, \text{Sets}], A \rightsquigarrow [-, A]$$

induces an equivalence between  $\underline{A}$  and the full subcategory of  $[\underline{A}(\alpha)^0, \text{Sets}]$  consisting of all functors  $\underline{A}(\alpha)^0 \rightarrow \text{Sets}$  which take  $\alpha$ -colimits in  $\alpha$ -limits ([13] 7.9, for the corresponding assertion

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with  $U_i, U_j \in M$  and  $\text{card}(J) < \beta > \text{card}(I)$  such that  $A$  is a retract of  $Y$  (cf [13] 7.6). (Recall that  $M$  is called regular if for every  $A \in \underline{A}$  there is a cokernel diagram  $K \rightrightarrows \coprod_{v \in V} U_v \rightarrow A$  with  $U_v \in M$ .)

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for  $\tilde{A}(\alpha)$  see [13] 9.10).

2.9 By the special adjoint functor theorem every colimit preserving functor between locally presentable categories has a right adjoint. By [13] 14.6 a limit preserving functor  $S : \underline{A} \rightarrow \underline{B}$  between locally presentable categories admits a left adjoint iff  $S$  has rank (cf. 2.1), i.e. iff  $S$  preserves  $\alpha$ -filtered colimits for some cardinal  $\alpha \geq \aleph_0$ .

2.10 Let  $\underline{U}$  be a small category and let  $\Sigma$  be a class of morphisms in  $[\underline{U}^0, \underline{\text{Sets}}]$ . Recall that a functor  $t : \underline{U} \rightarrow \underline{X}$  (resp.  $s : \underline{U}^0 \rightarrow \underline{X}$ ) is called  $\Sigma$ -cocontinuous (resp.  $\Sigma$ -continuous) if for every  $X \in \underline{X}$  and every  $\sigma \in \Sigma$  the map  $[\sigma, [t-, X]]$  (resp.  $[\sigma, [X, s-]]$ ) is bijective. If  $\underline{X}$  is cocomplete (resp. complete), then there is a tensor product bifunctor (resp. symbolic hom)

$$\begin{aligned} \otimes & : [\underline{U}^0, \underline{\text{Sets}}] \times [\underline{U}, \underline{X}] \longrightarrow \underline{X} \\ [-, -] & : [\underline{U}^0, \underline{\text{Sets}}] \times [\underline{U}^0, \underline{X}] \longrightarrow \underline{X} \end{aligned}$$

defined by

$$\begin{aligned} [R \otimes t, X] & \cong [R, [t-, X]] \\ [X, [R, s]] & \cong [R, [X, s-]] \end{aligned}$$

for all  $X \in \underline{X}$ ,  $R \in [\underline{U}^0, \underline{\text{Sets}}]$ ,  $t \in [\underline{U}, \underline{X}]$  and  $s \in [\underline{U}^0, \underline{X}]$ , cf Gabriel-Ulmer [13] 8.1. Hence  $t : \underline{U} \rightarrow \underline{X}$  (resp.  $s : \underline{U}^0 \rightarrow \underline{X}$ ) is  $\Sigma$ -cocontinuous (resp.  $\Sigma$ -continuous) iff  $\sigma \otimes t$  (resp.  $[\sigma, s]$ ) is an isomorphism for every  $\sigma \in \Sigma$ . The full subcategory of  $[\underline{U}, \underline{X}]$  consisting of all  $\Sigma$ -cocontinuous functors is denoted with  $Cc_{\Sigma}[\underline{U}, \underline{X}]$ . Likewise  $C_{\Sigma}[\underline{U}^0, \underline{X}]$  denotes the full subcategory of all  $\Sigma$ -continuous functors.

Examples for  $\Sigma$ -continuous (resp.  $\Sigma$ -cocontinuous) functors are sheaves (resp. cosheaves) with respect to a Grothendieck topology and functors which take a given class of colimits into limits (resp. colimits) etc, see § 6.

A class  $\Sigma$  of morphisms in  $[\underline{U}^0, \underline{\text{Sets}}]$  is called closed if

- 1)  $\Sigma$  contains all isomorphisms 2)  $\Sigma$  is closed under colimits  
 3) if  $\rho = \sigma\tau$  and two of the morphisms  $\rho, \sigma, \tau$  belong to  $\Sigma$ , then so does the third.

For instance, if  $T$  is a class of functors  $\underline{U} \rightarrow \underline{X}$ , then the class  $\Omega$  of all morphisms  $\omega$  such that  $\omega \circ t$  is an isomorphism for every  $t \in T$  is closed.

The closure  $\bar{\Sigma}$  of a class  $\Sigma$  is the smallest closed class containing  $\Sigma$ . Hence a  $\Sigma$ -cocontinuous functor  $\underline{U} \rightarrow \underline{X}$  is also  $\bar{\Sigma}$ -cocontinuous.

Let  $\Sigma$  be a class of morphisms in  $[\underline{U}^0, \underline{Sets}]$ , where  $\underline{U}$  is a small category, such that the codomains  $r\sigma, \sigma \in \Sigma$ , form a set (modulo equivalence). Then by [13] 8.11\*) the inclusion  $C_{\Sigma}[\underline{U}^0, \underline{Sets}] \hookrightarrow [\underline{U}^0, \underline{Sets}]$  has a left adjoint and a morphism  $\tau$  in  $[\underline{U}^0, \underline{Sets}]$  belongs to  $\bar{\Sigma}$  iff  $[\tau, t]$  is a bijection for every  $t \in C_{\Sigma}[\underline{U}^0, \underline{Sets}]$ .

2.11 A category  $\underline{A}$  is locally presentable iff there is a small category  $\underline{U}$  together with a set  $\Sigma$  of morphisms in  $[\underline{U}^0, \underline{Sets}]$  such that  $\underline{A} \cong C_{\Sigma}[\underline{U}^0, \underline{Sets}]$ , cf. [13] 8.5, 8.6 c). Moreover if  $\underline{B}$  is any locally presentable category and  $\underline{U}$  and  $\Sigma$  are as above, then  $C_{\Sigma}[\underline{U}^0, \underline{B}]$  is again locally presentable and

$$\pi(C_{\Sigma}[\underline{U}^0, \underline{B}]) \leq \sup_{\sigma \in \Sigma}^* (\pi(\underline{B}), \pi(d\sigma), \pi(r\sigma))$$

where  $d\sigma$  (resp.  $r\sigma$ ) denotes the domain (resp. codomain) of  $\sigma \in \Sigma$  and  $\sup^*(\ )$  denotes the least regular cardinal  $\geq \sup(\ )$ , cf. [13] 8.7.

2.12 Let  $\mathbb{T} = (T, u, \mu)$  be a triple in a locally presentable category  $\underline{A}$ . Then by [13] §10 the category of  $\mathbb{T}$ -algebras  $\underline{A}$  is locally presentable iff  $T$  has rank (2.1). Moreover if  $T$  has rank, then

$$\pi(\underline{A}^{\mathbb{T}}) \leq \sup \{ \pi(\underline{A}), \pi(T) \}$$

\*) The proofs of [13] 8.10 and 8.11 have a gap: On p.99 it is used that in  $\underline{A}_{\Sigma_2}$  every object has only a set of proper quotients. This may not be the case unless  $\underline{A}$  has additional properties. The easiest way out is to assume that  $\underline{A}$  is locally presentable ...